Dynamics of one impurity in the presence of dark solitons in atomic mixtures

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Abstract: We consider a system composed by two coupled Bose-Einstein condensates trapped in a quasi-1D trap. We consider a specific configuration in which one of the condensates is much more populated than the other one. In this setup, we study how the less populated condensate gets trapped in the presence of a dark soliton in the first condensate. Analytical and numerical solutions are obtained. Then, we consider the case of two dark solitons in the first condensate and study the tunnelling dynamics of the second component in the effective double-well potential induced by the dark solitons.

I. INTRODUCTION

In the last decade the stability and dynamics of dark solitons in Bose-Einstein condensates (BECs) have become an important subject that has motivated a large amount of studies. Solitons have been observed in a large variety of systems as optical fibers, magnetic films, plasmas, and wave guide arrays water or atomic Bose-Einstein condensates [1]. In our case, dark solitons are macroscopic structures in BECs, which consist of localized depressions in the density profile in a one-dimensional BEC. These structures emerge due to a precise balance between the nonlinear and dispersive effects in the atomic system [2]. These nonlinear effects are a direct consequence of the interactions between the atoms. Furthermore, in the case of dark solitons these interactions are repulsive. Another relevant property of dark solitons is that they can travel with constant speed while keeping their shape [1].

In this work we consider a system composed by two coupled Bose-Einstein condensates trapped in a quasi-1D trap. In particular we consider a specific configuration in which one of the condensates is much more populated. Our aim is to study how the less populated condensate behaves in the presence of dark solitons associated to the most populated component in the BEC. To this end, we will study the extreme case of one impurity in the presence of one or two solitons of the most populated component. The first case, with only one soliton, can be analytically studied. While the two soliton case requires the construction of a numerical code. The density profile of the two solitons resembles a double-well potential and we study the dynamics of the impurity tunnelling through the barrier between the two separated wells.

The present work is structured as follows: in Section II we describe the mean-field approximation of the two component system. In Section III we discuss the mathematical model for a single dark soliton in one component. In the Section IV we present the numerical algorithm to build two dark solitons in one component. Finally, in Section V we study the tunnelling dynamics of one impurity in the system of two dark solitons considered as a double-well static potential. A short summary and the main conclusions are presented in Section VI.

II. MEAN-FIELD DESCRIPTION

In the limit of zero temperature, the dynamics of a one-dimensional Bose-Einstein condensate with two different components, at the mean-field level, is described by the wave functions $\psi_1(x,t)$ and $\psi_2(x,t)$. They obey the following system of coupled non-linear differential equations, known as Gross-Pitaevskii (GP) equations [2]

$$i\hbar\frac{\partial\psi_1}{\partial t} = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2\right]\psi_1\,, \tag{1}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2\right]\psi_2\,, \tag{2}$$

where $\hbar$ is the Planck constant, m is the atomic mass of each species (which are taken equal) and g_{ij} is the strength of the different atomic interactions. We consider the homogeneous case. As the second component is much less populated we can assume that the first component is decoupled from the second one. In addition, we also neglect the inter-atomic interactions in the second component. Therefore, under these assumptions the system is described by the following two equations:

$$i\hbar\frac{\partial\psi_1}{\partial t} = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{11}|\psi_1|^2\right]\psi_1\,, \tag{3}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{12}|\psi_1|^2\right]\psi_2\,. \tag{4}$$

A. Stationary case

In this subsection we study the stationary case. Considering that the wave function of the condensate evolves in time according to [3]

$$\psi_1(x,t) = \psi_1(x)e^{-i\frac{\mu t}{\hbar}}\,, \tag{5}$$

where μ is the chemical potential, the GP time-independent equation reduces to

$$\mu\psi_1(x) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{11}|\psi_1(x)|^2\right]\psi_1(x)\,. \tag{6}$$

Notice that a simple constant wave function like, $\psi(x) = \sqrt{n}$, is a solution of the Eq. (6). In this case, the density is $|\psi(x)|^2 = n$ and the chemical potential is given by $\mu = g_{11}n$. Another possible solution known as dark soliton is given by

$$\psi_1(x) = \sqrt{n_o}\tanh\left(\frac{x\sqrt{mg_{11}n_o}}{\hbar}\right)\,, \tag{7}$$

where n_o is the background density [2, 4]. For this solution, the chemical potential is given by $\mu_1 = g_{11}n_o$, while the number of missing atoms from the background density due to the depression created by the wave function of Eq. (7) is given by

$$N = \int_{-\infty}^{\infty}(|\psi_1(x)|^2 - n_o)\mathrm{d}x = -2\hbar\sqrt{\frac{n_o}{mg_{11}}}\,, \tag{8}$$

which is in agreement with the results of reference [4].

III. ANALYTICAL SOLUTION

In this section we solve analytically the Gross-Pitaevskii equation of the second component Eq. (4), in the presence of the dark soliton of the first component Eq. (7) acting as a potential well.

The time-independent Gross-Pitaevskii equation corresponding to Eq. (4), assuming $\psi_2(x,t) = \psi_2(x)e^{-i\frac{\mu_2 t}{\hbar}}$, is given by

$$\mu_2\psi_2(x) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{12}|\psi_1(x)|^2\right]\psi_2(x)\,. \tag{9}$$

Taking the density profile of the dark soliton of Eq. (7) we arrive to the following equation

$$\mu_2\psi_2(x) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{12}n_o\tanh^2\left(\frac{x\sqrt{mg_{11}n_\infty}}{\hbar}\right)\right]\psi_2(x)\,. \tag{10}$$

A relevant quantity for the description of a soliton is the healing length of a soliton

$$\xi = \frac{\hbar}{\sqrt{mg_{11}n_o}}\,, \tag{11}$$

which measures the minimum spatial scale of density variations [1]. Introducing the healing length in the previous equation we obtain

$$\mu_2\psi_2(x) = \left[-\frac{\hbar^2}{2m}\partial_x^2 + g_{12}n_o\tanh^2\left(\frac{x}{\xi}\right)\right]\psi_2(x)\,. \tag{12}$$

Measuring the length in terms of the healing length, $\tilde{x} = \frac{x}{\xi}$, we can rewrite the time-independent Schödinger equation Eq. (12) as,

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\tilde{x}^2} + \frac{g_{12}}{g_{11}}\tanh^2(\tilde{x})\right]\psi_2(\tilde{x}) = \frac{\mu_2}{\mu_1}\psi_2(\tilde{x})\,. \tag{13}$$

A physical interpretation of Eq. (13) is to consider that the static dark soliton acts as a potential well to the second component,

$$V(\tilde{x}) = \frac{g_{12}}{g_{11}}\tanh^2(\tilde{x})\,. \tag{14}$$

It turns out that it is convenient to shift the energy potential such that $V(\tilde{x}) \to 0$, when $|\tilde{x}| \to \infty$. The shifted potential is defined by

$$V(\tilde{x}) = \frac{g_{12}}{g_{11}}(\tanh^2(\tilde{x}) - 1) = -\frac{g_{12}}{g_{11}}\mathrm{sech}^2(\tilde{x})\,, \tag{15}$$

which resembles the Pöschl-Teller Potential [5]. The equation to solve is the following

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\tilde{x}^2} - \frac{g_{12}}{g_{11}}\mathrm{sech}^2(\tilde{x})\right]\psi_2(\tilde{x}) = \frac{\mu_2}{\mu_1}\psi_2(\tilde{x})\,. \tag{16}$$

To solve this equation, we transform it to a Associated Legendre differential equation as [6, 7]

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\psi(x)\right] + \left(n(n+1) - \frac{m^2}{1-x^2}\right)\psi(x) = 0\,, \tag{17}$$

where $m, n \in \mathbb{R}$, which general solution is [6, 8]

$$\psi(x) = AP_n^m(x) + BQ_n^m(x)\,, \tag{18}$$

where $P_n^m(x)$ and $Q_n^m(x)$ are the Associated Legendre polynomials of the first and second type respectively.

The appropriate change of variable to transform Eq. (16) to a Legendre differential equation is $u = \tanh(\tilde{x})$, which allows to write the Eq. (16) as

$$\frac{d}{du}\left[(1-u^2)\frac{\partial\psi_2}{\partial u}\right] + \left[2\frac{\mu_2}{\mu_1}\frac{1}{1-u^2} + 2\frac{g_{12}}{g_{11}}\right]\psi_2 = 0\,. \tag{19}$$

Comparing Eq. (17) and Eq. (19), we can identify

$$n = \frac{1}{2}\left(\pm\sqrt{1 + 8\frac{g_{12}}{g_{11}}} - 1\right)\,, \tag{20}$$

$$\frac{\mu_2}{\mu_1} = \frac{-m^2}{2}\,. \tag{21}$$

The positive values of the index n define the bound states solutions of the Schrödinger time-independent equation as Associated Legendre polynomials, with $m = -n, -n+1, -n+2, \cdots < 0$,

$$\psi(\tilde{x}) = AP_n^m(\tanh(\tilde{x}))\,, \tag{22}$$

where A is a normalization constant, and boundary conditions are $\psi(-\infty) = 0$ and $\psi(\infty) = 0$.

For instance, if the ratio $\frac{g_{12}}{g_{11}}$ is equal to one, then the indexes of the Associated Legendre polynomial will be $n = 1$ and $m = -1$. And therefore the normalized wave function of the only bound state, which corresponds to the ground state, for a single particle of the second component is given by

$$\psi(\tilde{x}) = \frac{\sqrt{2}}{2}\text{sech}(\tilde{x})\,, \tag{23}$$

with energy $\frac{\mu_2}{\mu_1} = -\frac{1}{2}$.

The indexes of the Associated Legendre polynomials depend on the ratio $\frac{g_{12}}{g_{11}}$. In Fig.(1) we show the energies of the single-particle bound states, as a function of the ratio $\frac{g_{12}}{g_{11}}$, for different potential wells obtained as the result of using different interspecies interaction. We see that the more profound is the well of the potential described by Eq. (15) more bound states appear. In Fig.(2) we can see the single-particle density distribution of the second component, in different bound states. The number of bound states shown in each panel depends on the value of $\frac{g_{12}}{g_{11}}$. As expected, the density distributions are located around the center of the dark soliton which defines the minimum of the potential felt by the particle of the second component. Moreover, the number of zeros of the wave function increases as the excitation energy gets larger, and at the same time the particle gets less localized.

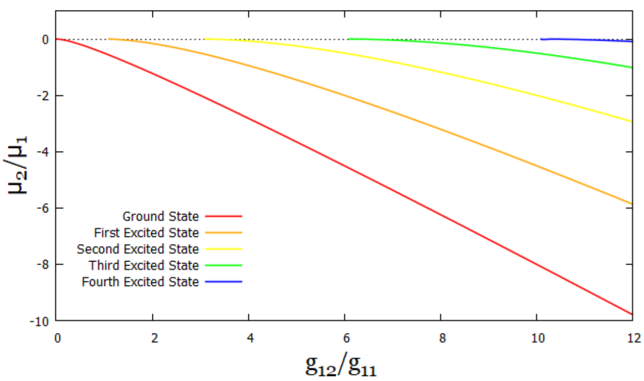

FIG. 1: Energies of the single-particle bound states of the second component for different potential wells obtained by varying the interspecies interaction.

IV. NUMERICAL METHODS

In this section, we discuss a numerical algorithm to face more difficult situations where simple analytical solutions do not exist. Our aim is to consider two dark solitons in the first component. These dark solitons are separated by $2d\ \xi$ and in a static configuration they will define a double-well potential for a particle of the second BEC component, under the same conditions of the previous sections. In this case, the Schrödinger equation for the impurity is given by

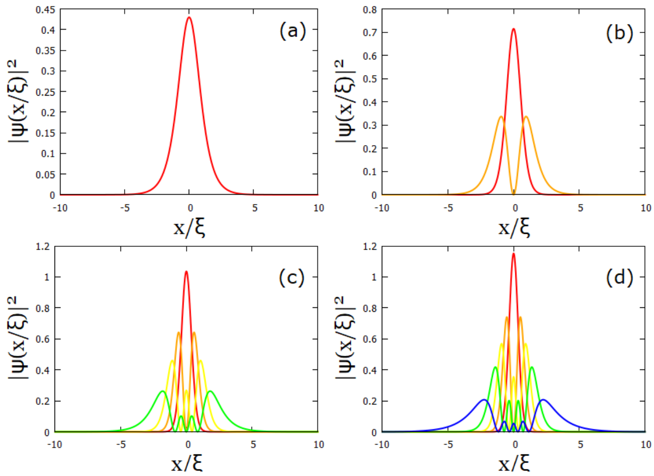

FIG. 2: Density profiles of each single-particle bound state for different potential wells. The red line describes the ground state, while the orange, yellow, green, and blue lines correspond to the first, second, third and fourth excitation respectively, in the case that they exist. Panel (a) is the case $n = 0.78$ ($g_{12}/g_{11} = 0.7$) with only one bound state. Panel (b) corresponds to $n = 1.88$ ($g_{12}/g_{11} = 2.7$) with two bound states. Panel (c) $n = 3.63$ ($g_{12}/g_{11} = 8.4$) with four bound states. Finally, panel (d) shows the results for $n = 4.42$ ($g_{12}/g_{11} = 12.0$) with five bound states.

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\tilde{x}^2} - \frac{g_{12}}{g_{11}}(\text{sech}^2(\tilde{x}-d) + \text{sech}^2(\tilde{x}+d))\right]\psi_2(\tilde{x}) =$$

$$= \frac{\mu_2}{\mu_1}\psi_2(\tilde{x})\,. \tag{24}$$

To solve this equation numerically, we have to consider the values of the wave function as a set of discretized points (the mesh points) to define a vector: $\psi = (\psi_0, \psi_1, \psi_2, \ldots, \psi_N)$. Imposing the boundary conditions at the walls of the box as $\psi_0 = 0$ and $\psi_N = 0$, and expressing the second derivative by means of a three point formula, we can write the Eq. (16) in a matrix way [9].

$$\begin{pmatrix} A_1 & B & 0 & \cdots & 0 & 0 \\ B & A_2 & B & \cdots & 0 & 0 \\ 0 & B & A_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{N-2} & B \\ 0 & 0 & 0 & \cdots & B & A_{N-1} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix} = \frac{\mu_2}{\mu_1} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{N-2} \\ \psi_{N-1} \end{pmatrix}, \tag{25}$$

where A_n and B are given by

$$A_n = \frac{1}{\Delta\tilde{x}^2} - \frac{g_{12}}{g_{11}}\left[\text{sech}^2(n\Delta\tilde{x}^2 - d) + \text{sech}^2(n\Delta\tilde{x}^2 + d)\right], \tag{26}$$

$$B = \frac{-1}{2\Delta\tilde{x}^2}\,. \tag{27}$$

Diagonalizing the tridiagonal matrix, Eq. (25), we can get the eigenvalues and the eigenvectors of the problem, $|M - \frac{\mu_2}{\mu_1}\mathbb{I}| = 0$. We will be mainly interested in the bound states, i.e., localized and with negative energy. To validate our numerical procedure, we have first checked that the analytical results for the case of having only one soliton are well reproduced.

The bound-state energies as a function of the distance between both wells are shown in Fig.(3) for four different values of the ratio $\frac{g_{12}}{g_{11}}$. Notice that for $d = 0$, one recovers the previous results obtained for the potential created by only one soliton described by a double ratio $\frac{g_{12}}{g_{11}}$. It is interesting to observe that for each double-well, the energies of the different bound states merge by couples, when the distance between the wells increases. As expected, the energies of the bound states saturate when the distance between the wells gets large, i.e., they become independent of the distance.

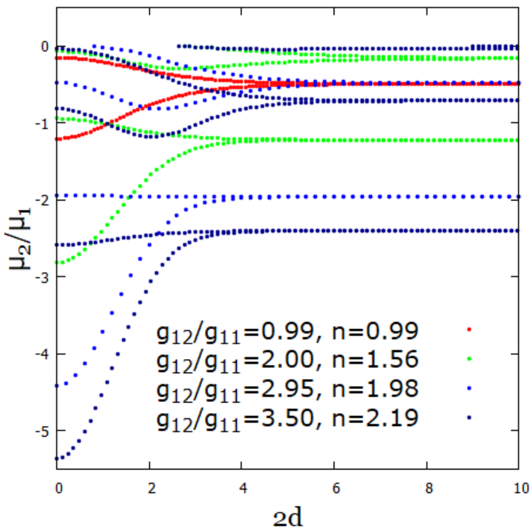

FIG. 3: Energies of the single-particle bound states in a double-well potential generated by the presence of two dark static solitons as a function of the separation between them in dimension of the healing length, for different values of the ratio g_{12}/g_{11}.

In Fig.(4) the shape of the potential as a function of the position is shown together with the energy of the different bound states. Panels (a) and (b) describe the potential for $g_{12}/g_{11} = 0.99$, for two different distances 2.24 ξ (a) and 10.43 ξ (b) between them. In panel (a) we see the two existing bound states well separated in energy. When the distance between the two wells increases, both states merge to a very similar value of the energy. In addition, this energy becomes independent of the distance when this distance increases. Panels (c) and (d) shows the corresponding results for $g_{12}/g_{11} = 2.95$. In this case, there are four bound states with different energies when the distance between the wells is 2.24 ξ. However, these states merge into two levels when the distance increases. Again the energy of these two levels becomes independent of the distance between the wells, when the distance increases.

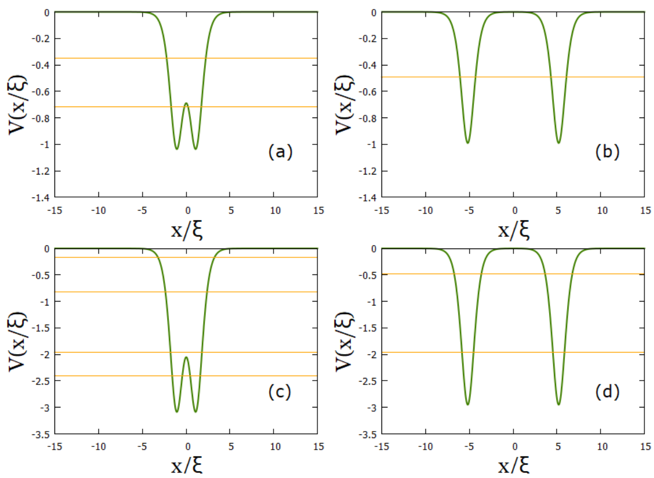

FIG. 4: Shape of the potential generated by two dark solitons. Panel (a) corresponds to a ratio of $g_{12}/g_{11} = 0.99$ with a distance 2.24 ξ between the wells. In this case, two bound states are obtained. Panel (b) shows the potential for the same ratio when the distance is $2d = 10.4$ ξ. The energies of the two states become almost degenerate. In the lower panels we have the case of $g_{12}/g_{11} = 2.95$. In panel (c), $2d = 2.24$ ξ where we see four bound states. Panel (d) corresponds to $2d = 10.4$ ξ, the four bound states merge into two levels.

V. TUNNELLING DYNAMICS

In this section we consider the tunnelling dynamics of the second component in the double-well created by two static dark solitons in the first component. To simplify the problem we consider a double-well static potential defined by the ratio of $g_{12}/g_{11} = 1$, where the two effective wells are separated by a distance 7 ξ. In this case there are only two bound states involved in the dynamics of the impurity. We prepare the second component in one of the effective wells to see how it tunnels to the other one. To this end, we compute the first two eigenstates of the second component, $\psi_0(\tilde{x})$ and $\psi_1(\tilde{x})$. Then, taking a linear combination $\frac{1}{\sqrt{2}}(\psi_0(\tilde{x}) + \psi_1(\tilde{x}))$ we have an initial state localized in the left well. Its dynamical evolution can be easily computed,

$$\psi(\tilde{x},t) = \frac{1}{\sqrt{2}}\left(\psi_o(\tilde{x})e^{\frac{-it\epsilon_o}{\hbar}} + \psi_1(\tilde{x})e^{\frac{-it\epsilon_1}{\hbar}}\right)\,, \tag{28}$$

where $\hbar$ is the Planck constant, the ϵ_i are the energies of the bound states and $\psi_i(\tilde{x})$ are the wave functions of the respective states.

In Fig.(5), we present the dimensionless time evolution of the probability density to illustrate how the wave packet goes from one well to the other.

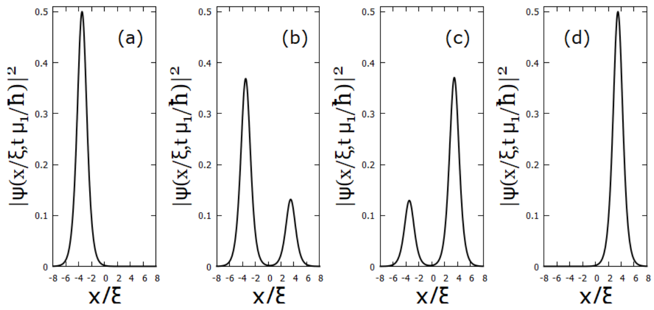

FIG. 5: Density profile of the wave packet for different times. Panel (a),(b),(c) and (d) are for $\tilde{t} = 0, 295.2, 568.8$ and 856.8 respectively, for the case where the separation of double wells is 7 ξ and $g_{12}/g_{11} = 1$.

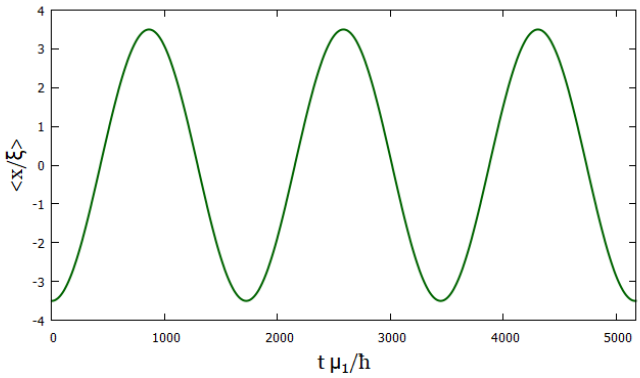

FIG. 6: Expected value of $\frac{x}{\xi}$ of a wave packet as a function of time in a double-well potential created by two static dark solitons separated by $2d = 7\ \xi$, characterized with a ratio $\frac{g_{12}}{g_{11}} = 1$.

Fig.(6) shows the expected value of $\frac{x}{\xi}$, denoted by $\langle \frac{x}{\xi} \rangle$, as a function of the dimensionless time variable $\tilde{t} = t\mu_1/\hbar$. One can see how the wave packet oscillates between the two wells. Moreover, we observe that the number of oscillations increases when the distance between the two wells decreases.

VI. SUMMARY AND CONCLUSIONS

We have considered a system of two quasi 1D Bose-Einstein condensates at zero temperature described by the mean-field coupled Gross-Pitaevskii equations. We have studied a configuration in which one of the condensates is much more populated than the other one, and thus neglect the backaction of the second condensate on the first one. In this case, we have studied the possible bound solutions of the second condensate in the presence of a dark soliton in the first one.

We have found analytic solutions for the problem in terms of the Associated Legendre polynomials. We can thus, obtain analytically the number of bound states and their energy as a function of the interspecies coupling constant. We have numerically confirmed our analytic solutions. With the numerical tools developed we have studied and interesting dynamical configuration in which we pin two dark solitons on the first component and consider the tunnelling dynamics of the second component trapped in the soliton depression of the first one. Analyzing how the wave packet oscillates from one well to the other one, we observe that the frequency of the oscillations increases as the distance between the two solitons decreases.

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