Analysis of locality on a quantum computer with Bell-type inequalities.

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Abstract: The violation of the first three of Bell's inequalities is tested on the 5-qubit IBM quantum computer. A hidden variable proposition is used to make sure we do not cherry-pick our results. To prove the inequalities we created an entangled state and measured it on different directions. The experimental data shows that there is a clear violation of the three inequalities. Furthermore, the data matches the quantum mechanics prediction more than the hidden variables one. Thus we conclude that the computer behaves as quantum mechanics dictates with some systematic error.

I. INTRODUCTION

In 1935 Einstein, Podolsky, and Rosen (EPR) published an article [1] where they concluded that quantum mechanics is an incomplete theory. Furthermore, they discussed the possibility that it should be complemented by a set of hidden variables that would determine the result of what we measure. In other words, quantum mechanics should be a statistical consequence of them [1, 2]. This theory has been a controversial topic over the years. Bell changed that with the article [3], where he imposed the condition of locality over the hidden variable theory. Which meant that, on that framework, for changes to occur they must interact within a reasonable distance. Due to quantum mechanics postulates, entanglement does not require locality. Bell exploited that property and identified an experimentally measurable expectation value that could be used to compare both approaches and then formulated the first inequality (1964) which revealed the limitations of the hidden variable formalism. Later on, Clauser, Horne, Shimony and Holt (CHSH) proposed a second inequality of the same nature (1969) [2]. Furthermore, a year later Bell derived a more general inequality [4] which was used in the first experiment by Aspect in 1982, using entangled photons [5].

The purpose of this work is to test and compare these three inequalities on a quantum computer. Moreover, we want to see how quantum mechanics and the hidden variable theories resemble to the experimental data. The IBM-qe [8] offers an open access five-qubit quantum computer. It represents an opportunity to experiment quantum properties with ease.

The structure of the paper is the following. In section II we present the mathematical tools to understand the expectation value derived by Bell. In section III we propose a realistic hidden variable equation so that we can compare with the quantum mechanical predictions. In section IV we present the three inequalities that we are going to test. The following sections V and VI explain the basics of quantum gates and then the process of the simulation. In Sect. VII we present the main result and discussion. Finally, in Sect. VIII we summarize the main conclusions of our work.

II. FORMULATION

Bell, in order to prove that quantum mechanics (QM) is complete and not controlled by a set of hidden variables (HV), formulated mathematically an statistical product which is incompatible with the QM predictions. In this section we explain its meaning. For the purpose of this explanation we consider a pair of spin one half particles that are the result of a disintegration of a spin zero particle. Both particles are entangled, because the measurement of one spin direction determines the result of the other. In this case the state of the system is given by the singlet state,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle). \tag{1}$$

To measure the spins we use two separated Stern Gerlach Magnets (SGM), on certain direction $\vec{a}$ and $\vec{b}$. The corresponding result is $+1$ or -1 depending on the relative position of the spin with respect to the direction of the SGM. The results of the measurements are given by the functions $A(\lambda, \vec{a}) = \pm 1$ and $B(\lambda, b) = \pm 1$ where λ is a possible hidden variable.

For the HV to work, two hypothesis must be stated, the first one is that we assume that the measurement of one particle does not alter the result of the other particle, in other words the interactions must be local (2). The other hypothesis is that, according to QM, if we measure in the same direction on both SGM one result must be the opposite of the other (3), because of the entanglement proposed before

$$(A_{\mathbf{a}} \cdot B_{\mathbf{b}})(\lambda) = A_{\mathbf{a}}(\lambda) \cdot B_{\mathbf{b}}(\lambda) \tag{2}$$

$$A(\vec{a}, \lambda) = -B(\vec{a}, \lambda). \tag{3}$$

If we assume that $\rho(\lambda)$ is the normalised probability distribution of the HV, the expectation value of the product

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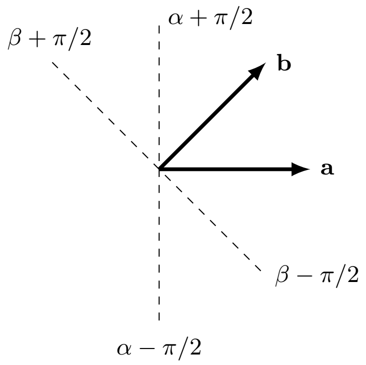

FIG. 1: Illustration of the different detection regions. The dashed lines mark the regions where the detectors give different results, see text for details.

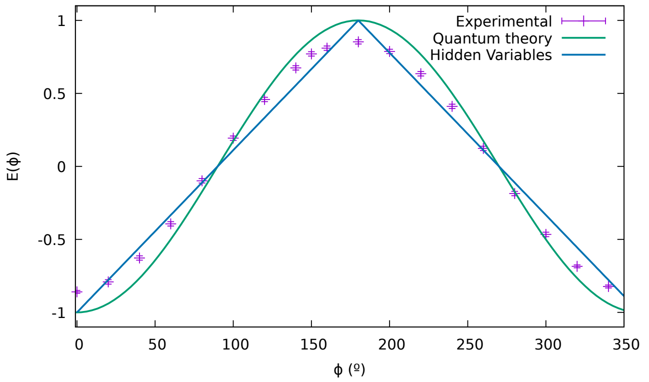

FIG. 2: Quantum mechanical and hidden variables prediction for $E(\vec{a},\vec{b})$ compared to the value measured experimentally on the IBMq system. In all cases, $\alpha = 0$, $\beta \in [0, 2\pi]$ and $\phi = |\alpha - \beta|$.

of both results is [3, 6, 7]

$$E(\vec{a},\vec{b}) = \int \rho(\lambda) A(\lambda,\vec{a}) B(\lambda,\vec{b}) d\lambda. \tag{4}$$

The corresponding QM result is, (where $\vec{a}$ and $\vec{b}$ are unitary vectors)

$$E_{QM}(\vec{a},\vec{b}) = \langle\Psi|\,\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}\,|\Psi\rangle = -\vec{a}\cdot\vec{b} = -\cos(\phi). \tag{5}$$

$\vec{\sigma}$ is the Pauli vector and ϕ is the angle between $\vec{a}$ and $\vec{b}$.

III. A REALISTIC HIDDEN VARIABLE PROPOSITION

In the following we present a concrete HV theory. We consider, following [3], the angle of polarisation of the spin as our HV λ and the direction of the spin as $\vec{\lambda}$. Now lets assume that if we measure a particle with λ automatically the other detector will get $\lambda + \pi$, due to the entanglement. Also we assume that the probability distribution is uniform for all its domain $[0, 2\pi)$.

Now lets assume that θ is the angle between $\vec{a}$ and $\vec{\lambda}$ (with $\vec{b}$ we use φ). If $\theta < \pi/2$ then $A = +1$ and if $\theta > \pi/2$ then $A = -1$, and correspondingly, $B = -1$ and $B = +1$. The functions that suit these conditions are

$$\begin{aligned} A(\vec{\lambda},\vec{a}) &= \mathrm{sgn}(\vec{a}\cdot\vec{\lambda}) \\ B(\vec{\lambda},\vec{b}) &= -\mathrm{sgn}(\vec{b}\cdot\vec{\lambda}) \qquad (6) \\ \rho(\lambda) &= \frac{1}{2\pi}. \qquad (7) \end{aligned}$$

With sgn the sign function, sgn(x) = x/|x|. This choice leads to four domain regions with results $A = B = 1$, $A = 1$ and $B = -1$, $A = -1$ and $B = 1$, and $A = B = -1$, see Fig. 1. From now on α and β are the angles of $\vec{a}$ and $\vec{b}$ respect to the x-axis, in this case,

$$\begin{aligned} E(\vec{a},\vec{b}) &= \frac{1}{2\pi}\int_0^{2\pi} A(\vec{a},\lambda)B(\vec{b},\lambda)d\lambda \\ &= -1 + \frac{2(\beta-\alpha)}{\pi}. \qquad (8) \end{aligned}$$

The equation (8) is for the particular case where β is larger than α and both are less than π, thus we state some symmetries. It must give us the same value if $\alpha > \beta$, also that $E(|\beta - \alpha|) = E(2\pi - |\beta - \alpha|)$, in other words, the result only depends on the angle between $\vec{a}$ and $\vec{b}$ [3, 7]. Hence taking $\phi = |\beta - \alpha|$ with $\alpha, \beta \in [0, 2\pi)$ we do the following changes

$$E(\phi) = \begin{Bmatrix} -1 + \frac{2\phi}{\pi} & \phi \le \pi \\ -1 + \frac{2(2\pi-\phi)}{\pi} & \phi > \pi \end{Bmatrix} \tag{9}$$

In Fig. 2 we compare the QM and HV predictions with experimental data obtained in the IBM-qe as explained in Sect. I. The QM and HV give different predictions, i.e. a cosine shape versus straight lines. The experimental results resemble the QM ones, but with a systematic discrepancy on the amplitude of the $\cos(\phi)$.

IV. BELL INEQUALITIES

It is not enough to compare both expectation values. Bell went one step further and devised a way to discern between QM and HV. In particular, he derived an inequality (10) assuming locality in the interaction between the particles and the fact that they are far apart when measured, [3, 6]

$$\left|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})\right| \le 1 + E(\vec{b},\vec{c}). \tag{10}$$

The second inequality (11) follows the same principles as the previous one, with the difference that accepts a non perfect correlation between two arbitrary directions, changing Bell's postulate. It is presented as $E(b, b') = 1 - \delta$ with $0 \leq \delta \leq 1$. That divides the domain of λ in two regions that causes $\Lambda^{\pm} = \{\lambda | A(\vec{b}', \lambda) = \pm B(\vec{b}, \lambda)\}$[2],

$$\left| E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) \right| \leq 2 - E(\vec{b}', \vec{b}) - E(\vec{b}', \vec{c}). \quad (11)$$

The third inequality (12) is commonly known as the CHSH inequality, it was derived by Bell on 1971. The only characteristic that differs from its relatives is that it accepts the possibility of more HV that act independently on every particle and make the experiment more realistic. It is assumed that the results detected are an averaged value of another HV $\bar{A}(\vec{a}, \lambda) = \int P(\mu_a) A(\vec{a}, \lambda, \mu_a) d\mu_a$.Where μ_a is a HV that acts only in the particle that interacts with the A detector. This assumption makes the results less correlated [4],

$$\left| E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{b}, \vec{a}') + E(\vec{a}', \vec{b}') \right| \leq 2. \quad (12)$$

For conceptual ease we will call each inequality, in order of appearance, BELL, CHSH and CHSHg. As the last one is somehow a generalisation of the second one. All of them make use of the expectation value computed above in Eq. (9).

Substituting (5) on any of the inequalities we shall find regions on the domain that violate them. Taking as an example the CHSHg we proceed in the following way. First using (5) and (12) we get

$$|\cos(\alpha + \beta + \gamma) - \cos(\alpha) - \cos(\beta) - \cos(\gamma)| \leq 2. \quad (13)$$

Since it is a three dimensional function, and even though we loose detail, we take $\gamma = 45^{\text{o}}$ and we take α and β as the x axis and the y axis respectively. Therefore we can plot the data (Fig. 3) showing that there is a region where the equation is violated. We choose $\gamma = 45^{\text{o}}$ because it goes through the absolute maximum which is at $\gamma = \alpha = \beta$. The left side of (12) gives $2\sqrt{2}$ as a result.

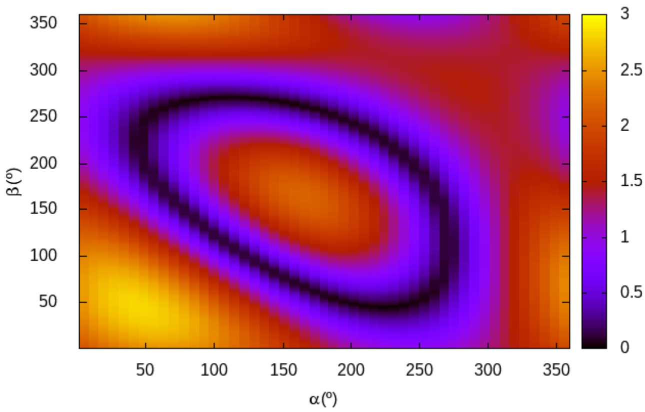

FIG. 3: Left hand side of the equation (13) where α is the angle between $\vec{a}$ and $\vec{b}$ and β is the angle between $\vec{b}$ and $\vec{a}'$. Finally $\gamma = 45^{\text{o}}$, which is the angle between $\vec{a}'$ and $\vec{b}'$. The region above two points is a violation of the inequality.

V. QUANTUM GATES

In this work, we will compare the theoretical QM and HV predictions outlined above with experimental results using the IBM-qe. The latter provides online access to a few qubits on a quantum computer. The idea is to build the entangled state, Eq. (1), and then perform the corresponding measurements entering in the inequalities presented above. First we must introduce some aspects of quantum computation. Lets consider a single qubit, it has two possible states $|0\rangle = |\uparrow\rangle$ and $|1\rangle = |\downarrow\rangle$. What differences from the bit of classical computation is the superposition of both states,

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,. \quad (14)$$

To measure the state of the qubit on any direction, the following operator is necessary

$$S_{\vec{n}} = \frac{\hbar}{2}\vec{\sigma} \cdot \vec{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}. \quad (15)$$

For simplicity, the constant is changed from $\frac{\hbar}{2}$ to 1 so we can have the eigenvalues as ± 1.

In order to predict the probabilities in the IBM quantum computer and prove that the QM predictions agree with the results, we need a series of single qubit gates that allow us to manipulate qbits maintaining its normalisation. The basic ones are the Pauli matrices,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (16)$$

Another useful one is the Hadamard gate, which creates a superposition state,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (17)$$

From the Pauli matrices we can get the rotation operators about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ axis, defined by the equations:

$$R_x(\theta) \equiv e^{-i\theta X/2} = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \quad (18)$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \quad (19)$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \quad (20)$$

Where θ is the angle of rotation around the axis. Finally, we need to introduce a two qubit gate, the CNOT. It affects two qubits at the same time. One acts as the control qubit and the other acts as the target. If the control qubit is $|1\rangle$ then the target qubit spin is flipped. This gate will provide us the entanglement,

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (21)$$

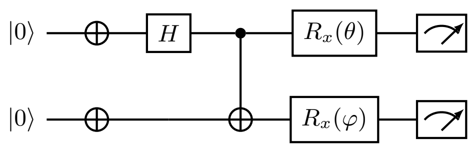

FIG. 4: Circuit implemented in the IBM-qe. The first gates produce the singlet state, then the rotations change the orientation of the spin. Finally, the two qubits are measured on the z basis. θ and φ are arbitrary angles.

	Exp.	QM	D
BELL	1.26 $\pm$0.03	1.5	0.47
CHSH	2.41 $\pm$0.04	2.82	0.41
CHSHg	2.36 $\pm$0.04	2.82	0.46

TABLE I: Comparison of the three inequalities at the point where the violation is maximum. The first column corresponds to the experimental data, the second column is the QM prediction and the third one corresponds to the discrepancy.

VI. EXPERIMENTAL TEST ON THE IBM-QE

On the IBM-qe we built a circuit to produce the singlet state (1) and then perform the measurements on the Z basis to measure the products entering in the inequalities. The circuit can be separated in three parts: production of the single state, which is an entangled state, rotation and measurement [9].

We start with both qubits on $|0\rangle$ then we flip both of them to $|1\rangle$ with the gate NOT (in the quantum computer acts as an X gate), after that we apply a Hadamard gate to the first qubit to create superposition of states $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then we apply the CNOT gate with the first qubit as a control and the second as the target. With that we create the entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$. Up until here we get to the entanglement part. Now the rotation part takes place,

$$|\Phi\rangle = R_x(\theta) \otimes R_x(\varphi)\,|\Psi\rangle = \tag{22}$$
$$\left[i \sin\left(\frac{\varphi-\theta}{2}\right) \frac{|00\rangle - |11\rangle}{\sqrt{2}} + \cos\left(\frac{\varphi-\theta}{2}\right) \frac{|10\rangle - |01\rangle}{\sqrt{2}} \right].$$

Finally, we measure the state (22). To assure that we have done the theory correctly, we check that the expected value is the same as (5). As expected, it is correct.

$$E_{QM} \equiv \langle S_z \rangle = \sin^2(\phi/2) - \cos^2(\phi/2) = -\cos\phi \tag{23}$$

where $\phi = \theta - \varphi$ acts as the angle between the two orientations.

The IBM makes 20000 shots of the circuit and registers the frequency of the resulting states ($|00\rangle$, $|10\rangle$, $|01\rangle$, $|11\rangle$). How can we match the results extracted with the theoretical expectation value? We can turn these frequency shots into probabilities by dividing by the total of shots $P_{ij} = N_{ij}/N_{tot}$ and substitute the theoretical probabilities that conform (23) with the experimental data. Using (22), the math is the following

$$\begin{aligned} \langle S_z \rangle &= \lambda_{00} P_{00} + \lambda_{10} P_{10} + \lambda_{01} P_{01} + \lambda_{11} P_{11} \\ &= P_{00} - P_{10} - P_{01} + P_{11} \end{aligned} \tag{24}$$
$$= \frac{1}{2}\left[\sin^2(\phi/2) - \cos^2(\phi/2) - \cos^2(\phi/2) + \sin^2(\phi/2)\right]. \tag{25}$$

Where λ_{ij} is the eigenvalue of each eigenstate, (24) is for the experimental data and (25) it is what is deduced of the theoretical probabilities.

VII. RESULTS

Now we shall discuss the results obtained. In order to analyse the data we did the same as on Fig. 3. However, instead of fixing only one angle, we leave one free. This method gives us enough information to see the violations. We take BELL and CHSH and we put all the expectation values to the left of the equation. With the CHSHg there is no need to change anything. Therefore, to plot the theoretical HV or QM prediction, we just replace the expectation value with (2) and (5) respectively.

We have a Binomial type distribution, due to the fact that we counted the number of successes in a sequence of independent experiments n with a yes-no question. Therefore the error associated is $\sigma_i = \sqrt{n p_i (1 - P_i)}$. The following expansion of the error of the expectation value leads to $\delta = (\sigma_{00} + \sigma_{01} + \sigma_{10} + \sigma_{11})/n$. In conclusion, the error oscillates between 0.02 and 0.04. As we see on the graphical representations the discrepancies are $> 3\delta$, thus we must assume some systematic error that we cannot control.

Notice that on Table I the statistical error on BELL is less than the others, due to the fact that it has less expectation values, although it is not significant because they are so small.

To plot such representations we performed over 100 independent experiments on which we changed the angle of measurement every time.

From a general standpoint, we can see that the QM prediction matches, on a more accurate way, the behaviour of the experimental data. It does, indeed, violate the inequality, whilst the HV prediction cannot predict the behavior once surpassed the threshold stipulated by the equations (10), (11) and (12). However, the HV plots can show where the violation takes place, it remains constant on the limit. On Figs. 6 and 7 the region where the inequality is violated is better predicted with the HV formalism.

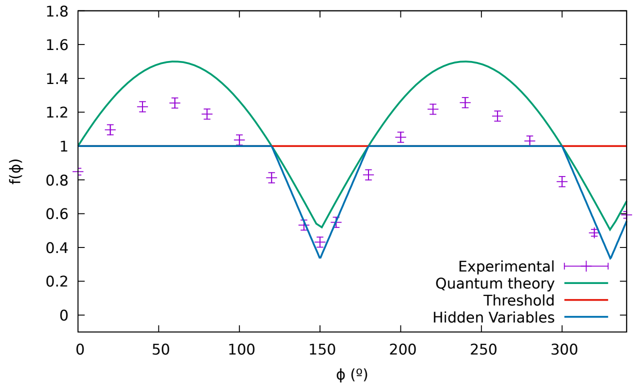

FIG. 5: Plot of the left side of the BELL inequality $f(\phi) = \left|E(\theta_{\vec{a},\vec{b}}) - E(\theta_{\vec{a},\vec{c}})\right| - E(\theta_{\vec{b},\vec{c}})$. Angle configuration: $\theta_{\vec{a},\vec{b}} = 60$, $\theta_{\vec{b},\vec{c}} = \phi$ and $\theta_{\vec{a},\vec{c}} = \phi + \theta_{\vec{a},\vec{b}}$. Thirty-nine runs were made on the computer.

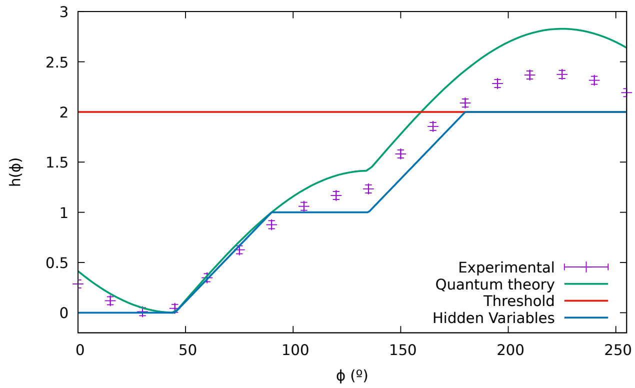

FIG. 6: Plot of the left side of the CHSH $h(\phi) = \left|E(\theta_{\vec{a},\vec{b}}) - E(\theta_{\vec{a},\vec{c}})\right| + E(\theta_{\vec{b}',\vec{b}}) + E(\theta_{\vec{b}',\vec{c}})$. Angle configuration is: $\theta_{\vec{a},\vec{b}} = 45$, $\theta_{\vec{b},\vec{b}'} = 225$, $\theta_{\vec{b}',\vec{c}} = \phi$ and $\theta_{\vec{a},\vec{c}} = \phi + 270$. Thirty-seven runs were made on the computer.

VIII. SUMMARY AND CONCLUSIONS

In this work we have studied three of the Bell inequalities and have tested tested on a quantum computer (IBM-qe). We have simulated the disintegration of a zero spin particle into two half-spin particles that are entangled. This experiment is made to prove that local theories, such as the HV proposition, lack of resources to predict the quantum behaviour. With that said, we have proven successfully that the IBM quantum computer behaves as QM anticipates. Nevertheless, as we fail to separate both qubits we can expect a HV pattern as they may influence each other.

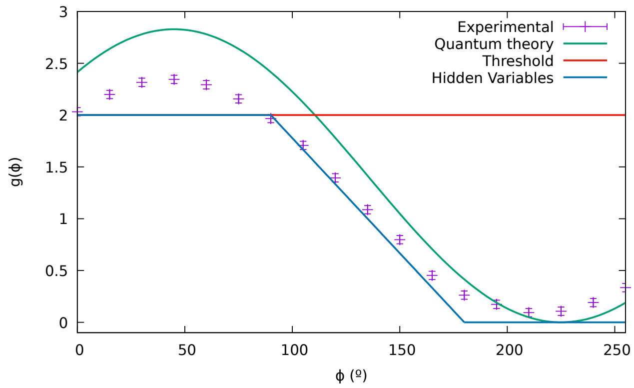

FIG. 7: Plot of the left side of the CHSHg $g(\phi) = \left|E(\theta_{\vec{a},\vec{b}}) - E(\theta_{\vec{a},\vec{b}'}) + E(\theta_{\vec{b},\vec{a}'}) + E(\theta_{\vec{a}',\vec{b}'})\right|$. Angle configuration: $\theta_{\vec{a},\vec{b}} = \theta_{\vec{b},\vec{a}'} = 45$, $\theta_{\vec{a}',\vec{b}'} = \phi$ and $\theta_{\vec{a},\vec{b}'} = 90 + \phi$. Thirty-seven runs were made on the computer.

Acknowledgments

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[1] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Complete?* Phys. Rev. 47, 777 (1935).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt , *Proposed Experiment to Test Local Hidden-Variable Theories*. Phys. Rev. Lett. 23, 880 (1969).
[3] J.S.Bell, *On the Einstein Podolsky Rosen paradox* Physics Physique Fizika 1, 195 (1964).
[4] J. S. Bell, Introduction to the hidden variable question, Proceedings of the International School of Physics 'Enrico Fermi', Course IL, Foundations of Quantum Mechanics 171-81 (1970).
[5] A. Aspect, P. Grangier and G. Roger, *Experimental Realisation of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities* Phys. Rev. Lett. 49, 91 (1982).
[6] Gennaro Auletta, *Foundations and Interpretation of Quantum Mechanics*, (World Scientific Publishing Co. Pte. Ltd., Singapore 2001 1rst. ed.).
[7] D. Dehlinger, M. W. Mitchell, *Entangled photons, nonlocality and Bell inequalities in the undergraduate laboratory* (2002).
[8] IBM. IBM Quantum Experience. Retrieved from https://quantum-computing.ibm.com (2021).
[9] D. García-Martín, G. Sierra. *Five Experimental Tests on the 5-Qbit IBM Quantum Computer*. J. Appl. Math. 6, 1460 (2018).